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ORIGINAL RESEARCH

Solvability of a coupled system of nonlinear fractional differential equations with fractional integral conditions

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Abstract In this paper, by applying the Schauder fixed point theorem, the Leray–Schauder nonlinear alternative and the Banach contraction principle, we establish some sufficient conditions for the existence and uniqueness of solutions for a coupled system of nonlinear fractional differential equations with fractional integral conditions, involving the Caputo fractional derivative. Some examples are given to illustrate our results.

Keywords Coupled system · Fractional integral conditions · Caputo fractional derivative · Fixed point theorem

Mathematics Subject Classification 26A33 · 34B18

1 Introduction

We consider a coupled system of nonlinear fractional differential equations (FDE for short of the form) with fractional integral conditions

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$$\begin{cases} {}^C D_{0+}^{\alpha_1} u(t) = f_1(t, u(t), v(t), {}^C D_{0+}^{\rho_1} u(t), {}^C D_{0+}^{\rho_2} v(t)), t \in [0, 1], \\ {}^C D_{0+}^{\alpha_2} v(t) = f_2(t, u(t), v(t), {}^C D_{0+}^{\rho_1} u(t), {}^C D_{0+}^{\rho_2} v(t)), t \in [0, 1], \\ u(\xi_1) = 0, u(1) = I_{0+}^{\theta_1} u(\eta_1), \\ v(\xi_2) = 0, v(1) = I_{0+}^{\theta_2} v(\eta_2), \end{cases} \tag{1.1}$$

where ${}^C D_{0+}^{\alpha_i}$ and ${}^C D_{0+}^{\rho_i}$ denote the Caputo fractional derivative, $I_{0+}^{\theta_i}$ denotes Riemann–Liouville fractional integral, $1 < \alpha_i < 2$, $f_i \in C([0, 1] \times \mathbb{R}^4, \mathbb{R})$, $0 < \rho_i < 1$, $0 \leq \xi_i < 1$, $0 \leq \eta_i \leq 1$, $\theta_i > 0$, $i = 1, 2$.

Fractional differential equations arise in many engineering and scientific disciplines such as the mathematical modeling of systems and processes in the fields of physics, chemistry, electrodynamics of complex medium, control theory, etc. We refer the reader to see [1–4]. Fractional differential equations are also regarded as a better tool for the description of hereditary properties of various materials and processes than the corresponding integer order differential equations. With this advantage, the subject of fractional differential equations is gaining much importance and attention. For some recent development on the topic, see [5–14] and the references therein. Recently, Guezane-Lakoud and Khaldi [15] investigated the existence and uniqueness of solution for a fractional boundary value problem with fraction integral condition

$$\begin{cases} {}^C D_{0+}^{q} u(t) = f(t, u(t), {}^C D_{0+}^{\sigma} u(t)), \quad t \in (0, 1), \\ u(0) = 0, u'(1) = I_{0+}^{\sigma} u(1), \end{cases} \tag{1.2}$$

where $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is a given continuous function, $1 < q < 2$, $0 < \sigma < 1$. The results allow the integral condition to depend on the fractional integral $I_{0+}^{\sigma} u$ which leads to extra difficulties.

On the other hand, the study of a coupled system of fractional order is also very significant because this kind of system can often occur in various applications. There are a large number of papers dealing with the solvability of coupled systems of nonlinear fractional differential equations. For details, see [16–23] and the references cited therein. In [24], the authors studied a coupled system of nonlinear fractional differential equations with three-point boundary conditions

$$\begin{cases} D^{\alpha} u(t) = f(t, v(t), D^{p} v(t)), \quad t \in (0, 1), \\ D^{\beta} v(t) = g(t, u(t), D^{q} u(t)), \quad t \in (0, 1), \\ u(0) = 0, u(1) = \gamma u(\eta), \\ v(0) = 0, v(1) = \gamma v(\eta), \end{cases} \tag{1.3}$$

where $1 < \alpha, \beta < 2$, $p, q, \gamma > 0$, $0 < \eta < 1$, $\alpha - q \geq 1$, $\beta - p \geq 1$, $\gamma \eta^{\alpha-1} < 1$, $\gamma \eta^{\beta-1} < 1$. D is the standard Riemann–Liouville fractional derivative and $f, g : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ are given continuous functions. By applying the Schauder fixed point theorem, an existence result which improved the work in [16] was proved.

In [25], by applying some standard fixed point theorems, the existence results are obtained for the coupled system of fractional differential equations with nonlocal integral boundary conditions

$$\begin{cases} {}^{C}D^{\alpha}u(t) = f(t, v(t), D^{p}v(t)),\ {}^{C}D^{\beta}v(t) = g(t, u(t), D^{q}u(t)),\ \ t \in (0,1), \\ au'(0) + u(\eta_1) = \int_0^1 \phi(s, v(s))ds,\ bu'(1) + u(\eta_2) = \int_0^1 \psi(s, v(s))ds, \\ cv'(0) + v(\xi_1) = \int_0^1 \varphi(s, u(s))ds,\ dv'(1) + v(\xi_2) = \int_0^1 \rho(s, u(s))ds, \end{cases} \tag{1.4}$$

where $1 < \alpha, \beta < 2, 0 < p, q < 1$ and $\alpha - p - 1 \geq 0, \beta - q - 1 \geq 0.0 \leq \eta_1 < \eta_2 \leq 1, 0 \leq \xi_1 < \xi_2 \leq 1. f, g, \phi, \psi, \varphi, \rho,$ are given functions satisfying some assumptions. The form of the integral the authors consider here is quite general, which involves some of known results.

In [26], the existence and uniqueness of solutions for a boundary value problem of first-order fractional differential equations with Rieman-Liouville integral boundary conditions is studied

$$\begin{cases} {}^{C}D^{\alpha}_{0+}u(t) = f(t, u(t), v(t)),\ \ t \in [0,1], \\ {}^{C}D^{\beta}_{0+}v(t) = g(t, u(t), v(t)),\ \ t \in [0,1], \\ u(0) = \gamma I^{p}_{0+}u(\eta),\ \ 0 < \eta < 1, \\ v(0) = \delta I^{q}_{0+}v(\xi),\ \ 0 < \xi < 1, \end{cases} \tag{1.5}$$

where ${}^{C}D^{\alpha}_{0+}$ and ${}^{C}D^{\beta}_{0+}$ denote the Caputo fractional derivative, I^{p}_{0+}, I^{q}_{0+} denote Riemann–Liouville fractional integral, $0 < \alpha, \beta < 1$, $f, g \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$, and $p, q, \gamma, \delta \in \mathbb{R}$.

From above, we can see a fact, although the coupled system of fractional boundary value problems have been investigated by some authors, FDE (1.1) are seldom considered and present more general argument. The main features of the present paper are follows: First, compared with [16–26], the system we discuss here is coupled not only in the differential system but also through the nonlinear terms f_1, f_2, which involved two unknown functions u, v and the fractional derivative of unknown functions u, v, and the case is more complicated and difficult than the nonlinear terms involved only a unknown function and the fractional derivative of a unknown function. Secondly, compared with the above mentioned documents, what we discuss here allow the integral condition to depend on the fractional order integral $I^{\sigma}_{0+}u$ which covers the multi-point boundary conditions and integer order integral boundary conditions. It is worth mentioning that integral boundary conditions are encountered in population dynamics, blood flow models, cellular systems, heat transmission, plasma physics, etc.. They come up when values of the function on the boundary are connected to its value inside the domain. Sometimes, it is better to impose integral conditions because they lead to more precise measure than the local conditions. Furthermore, what we discuss here allow $u(\xi_i) = 0, 0 \leq \xi_i < 1$ which is a more general condition, instead of $u(0) = 0$ in the literature [15–22,24]. Finally, compared with [16,18,20,23–26], the methods and some growth conditions (see Theorem 3.2) what we use here are of some difference.

The rest of the paper is organized as follows. In Sect. 2, we present preliminaries and several lemmas. In Sect. 3, solvability of nonlinear FDE (1.1) is formulated and proved by using a variety of methods. In Sect. 4, some examples are given to demonstrate the main results.

2 Preliminaries

For the convenience of readers, in this section, we present some definitions and lemmas. More information on fractional calculus can be found in, for example, [2,3].

Definition 2.1 If $f \in AC^n([a,b],\mathbb{R})$ and $\alpha > 0$, then the Caputo fractional derivative of order α is given by

$$^C D^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} \mathrm{d}t,$$

where $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of number α.

Definition 2.2 If $f \in C([a,b],\mathbb{R})$ and $\alpha > 0$, then the Riemann–Liouville fractional integral of order α is given by

$$I^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} \mathrm{d}t.$$

Lemma 2.1 *For $\alpha > 0$, the fractional differential equation $^C D^{\alpha}_{0+} u(t) = 0$ has a general solution*

$$u(t) = c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, and $n = [\alpha]+1$.

Lemma 2.2 *Let $p > q \geq 0$, $g(t) \in L^1(a,b)$. For any $t \in [a,b]$, then*

$$I^p_{0+} I^q_{0+} g(t) = I^{p+q}_{0+} g(t) = I^q_{0+} I^p_{0+} g(t), {}^C D^p_{0+} I^p_{0+} g(t) = g(t), {}^C D^q_{0+} I^p_{0+} g(t) = I^{p-q}_{0+} g(t).$$

Let $C[0,1]$ denotes the space of all continuous functions. Set $U_\alpha = \{u(t) | u(t) \in C[0,1]$ and $^C D^\alpha u(t) \in C[0,1]\}$. In order to prove our main results, we need the following lemmas.

Lemma 2.3 *Let $1 < \alpha < 2$, $0 \leq \xi < 1$, $0 \leq \eta \leq 1$, $\theta > 0$, $\Delta_2 - \Delta_1 \xi \neq 0$. For any $y(t) \in C[0,1]$, then the unique solution of the fractional boundary value problem*

$$\begin{cases} ^C D^{\alpha}_{0+} u(t) = y(t), \ t \in [0,1], \\ u(\xi) = 0, u(1) = I^{\theta}_{0+} u(\eta), \end{cases} \tag{2.1}$$

is given in U_α by

$$u(t) = \Delta_3(\xi - t) I^{\alpha}_{0+} y(1) + \Delta_3(t-\xi) I^{\alpha+\theta}_{0+} y(\eta) + \Delta_3(\Delta_1 t - \Delta_2) I^{\alpha}_{0+} y(\xi) + I^{\alpha}_{0+} y(t), \tag{2.2}$$

where $\Delta_1 = (1 - \frac{\eta^\theta}{\Gamma(\theta+1)})$, $\Delta_2 = (1 - \frac{\eta^{\theta+1}}{\Gamma(\theta+2)})$, $\Delta_3 = \frac{1}{\Delta_2 - \Delta_1 \xi}$.

Proof Applying Lemma 2.1, the equation ${}^C D^{\alpha}_{0+} u(t) = y(t)$ in (2.1) means

$$u(t) = c_1 + c_2 t + I^{\alpha}_{0+} y(t). \tag{2.3}$$

Using the condition $u(\xi) = 0$ and (2.3), we have

$$c_1 + c_2 \xi = -I^{\alpha}_{0+} y(\xi). \tag{2.4}$$

Using Lemma 2.2 and (2.3), we have

$$I^{\theta}_{0+} u(t) = c_1 \frac{t^{\theta}}{\Gamma(\theta+1)} + c_2 \frac{t^{\theta+1}}{\Gamma(\theta+2)} + I^{\alpha+\theta}_{0+} y(t).$$

Moreover, the fractional integral condition $u(1) = I^{\theta}_{0+} u(\eta)$ leads to

$$c_1 + c_2 + I^{\alpha}_{0+} y(1) = c_1 \frac{\eta^{\theta}}{\Gamma(\theta+1)} + c_2 \frac{\eta^{\theta+1}}{\Gamma(\theta+2)} + I^{\alpha+\theta}_{0+} y(\eta),$$

that is

$$\Delta_1 c_1 + \Delta_2 c_2 = I^{\alpha+\theta}_{0+} y(\eta) - I^{\alpha}_{0+} y(1). \tag{2.5}$$

Combining (2.4) with (2.5), we obtain

$$\begin{aligned} c_1 &= \Delta_3[\xi I^{\alpha}_{0+} y(1) - \Delta_2 I^{\alpha}_{0+} y(\xi) - \xi I^{\alpha+\theta}_{0+} y(\eta)], \\ c_2 &= \Delta_3[I^{\alpha+\theta}_{0+} y(\eta) + \Delta_1 I^{\alpha}_{0+} y(\xi) - I^{\alpha}_{0+} y(1)]. \end{aligned}$$

Substituting c_1 and c_2 to (2.3), we obtain (2.2). The proof is completed.

Let $X = \{u(t) | u(t) \in C[0,1]$ and ${}^C D^{\rho_1} u(t) \in C[0,1]\}$ be a Banach space endowed with the norm $\|u\|_X = \max_{t\in J} |u(t)| + \max_{t\in J} |{}^C D^{\rho_1} u(t)|$, and $Y = \{v(t) | v(t) \in C[0,1]$ and ${}^C D^{\rho_2} v(t) \in C[0,1]\}$ be a Banach space endowed with the norm $\|v\|_Y = \max_{t\in J} |v(t)| + \max_{t\in J} |{}^C D^{\rho_2} v(t)|$. The product space $(X \times Y, \|(u,v)\|_{X\times Y})$ is also a Banach space with the norm $\|(u,v)\|_{X\times Y} = \|u\|_X + \|v\|_Y$.

Define the operator $T : X \times Y \to X \times Y$ by

$$T(u,v)(t) = (T_1(u,v)(t), T_2(u,v)(t)), \tag{2.6}$$

where

$$\begin{aligned} T_i(u,v)(t) = {} & \Delta_{i3}(\xi_i - t) I^{\alpha_i}_{0+} f_i(1, u(1), v(1), {}^C D^{\rho_1} u(1), {}^C D^{\rho_2} v(1)) \\ & + \Delta_{i3}(t - \xi_i) I^{\alpha_i+\theta_i}_{0+} f_i(\eta_i, u(\eta_i), v(\eta_i), {}^C D^{\rho_1} u(\eta_i), {}^C D^{\rho_2} v(\eta_i)) \\ & + \Delta_{i3}(\Delta_{i1} t - \Delta_{i2}) I^{\alpha_i}_{0+} f_i(\xi_i, u(\xi_i), v(\xi_i), {}^C D^{\rho_1} u(\xi_i), {}^C D^{\rho_2} v(\xi_i)) \\ & + I^{\alpha_i}_{0+} f_i(t, u(t), v(t), {}^C D^{\rho_1} u(t), {}^C D^{\rho_2} v(t)), \ i = 1, 2, \end{aligned}$$

and

$$\Delta_{i1}=\left(1-\frac{\eta_i^{\theta_i}}{\Gamma(\theta_i+1)}\right),\ \Delta_{i2}=\left(1-\frac{\eta_i^{\theta_i+1}}{\Gamma(\theta_i+2)}\right),$$
$$\Delta_{i3}=\frac{1}{\Delta_{i2}-\Delta_{i1}\xi_i},\ \Delta_{i2}-\Delta_{i1}\xi_i\neq 0,\quad i=1,2.$$

□

Remark 2.1

$$\begin{aligned}T_i(u,v)'(t)=&-\Delta_{i3}I_{0+}^{\alpha_i}f_i(1,u(1),v(1),{}^C D^{\rho_1}u(1),{}^C D^{\rho_2}v(1))\\&+\Delta_{i3}I_{0+}^{\alpha_i+\theta_i}f_i(\eta_i,u(\eta_i),v(\eta_i),{}^C D^{\rho_1}u(\eta_i),{}^C D^{\rho_2}v(\eta_i))\\&+\Delta_{i3}\Delta_{i1}I_{0+}^{\alpha_i}f_i(\xi_i,u(\xi_i),v(\xi_i),{}^C D^{\rho_1}u(\xi_i),{}^C D^{\rho_2}v(\xi_i))\\&+I_{0+}^{\alpha_i-1}f_i(t,u(t),v(t),{}^C D^{\rho_1}u(t),{}^C D^{\rho_2}v(t)),\ i=1,2.\end{aligned}$$

Set $U_{\alpha_1}=\{u(t)|u(t)\in C[0,1]\text{ and }{}^C D^{\alpha_1}u(t)\in C[0,1]\}$ and $U_{\alpha_2}=\{v(t)|v(t)\in C[0,1]\text{ and }{}^C D^{\alpha_2}v(t)\in C[0,1]\}$.

Lemma 2.4 *Let $f_1,f_2\in C([0,1]\times\mathbb{R}^4,\mathbb{R})$. Then $(u,v)\in U_{\alpha_1}\times U_{\alpha_2}$ is a solution of FDE (1.1) if and only if $(u,v)\in X\times Y$ is a solution of the operator equations $T(u,v)=(u,v)$.*

Proof Let $(u,v)\in U_{\alpha_1}\times U_{\alpha_2}$ be a solution of FDE (1.1). Applying Lemma 2.3 and (2.6), we can obtain immediately $(u,v)\in X\times Y$ is a solution of the operator equations $T(u,v)=(u,v)$. Conversely, let $(u,v)\in X\times Y$ is a solution of the operator equations $T(u,v)=(u,v)$. That is,

$$\begin{aligned}u(t)=&\ \Delta_{13}(\xi_1-t)I_{0+}^{\alpha_1}f_1(1,u(1),v(1),{}^C D^{\rho_1}u(1),{}^C D^{\rho_2}v(1))\\&+\Delta_{13}(t-\xi_1)I_{0+}^{\alpha_1+\theta_1}f_1(\eta_1,u(\eta_1),v(\eta_1),{}^C D^{\rho_1}u(\eta_1),{}^C D^{\rho_2}v(\eta_1))\\&+\Delta_{13}(\Delta_{11}t-\Delta_{12})I_{0+}^{\alpha_1}f_1(\xi_1,u(\xi_1),v(\xi_1),{}^C D^{\rho_1}u(\xi_1),{}^C D^{\rho_2}v(\xi_1))\\&+I_{0+}^{\alpha_1}f_1(t,u(t),v(t),{}^C D^{\rho_1}u(t),{}^C D^{\rho_2}v(t)),\\v(t)=&\ \Delta_{23}(\xi_2-t)I_{0+}^{\alpha_2}f_2(1,u(1),v(1),{}^C D^{\rho_1}u(1),{}^C D^{\rho_2}v(1))\\&+\Delta_{23}(t-\xi_2)I_{0+}^{\alpha_2+\theta_2}f_2(\eta_1,u(\eta_1),v(\eta_1),{}^C D^{\rho_1}u(\eta_1),{}^C D^{\rho_2}v(\eta_1))\\&+\Delta_{23}(\Delta_{21}t-\Delta_{22})I_{0+}^{\alpha_2}f_2(\xi_1,u(\xi_1),v(\xi_1),{}^C D^{\rho_1}u(\xi_1),{}^C D^{\rho_2}v(\xi_1))\\&+I_{0+}^{\alpha_2}f_2(t,u(t),v(t),{}^C D^{\rho_1}u(t),{}^C D^{\rho_2}v(t)).\end{aligned}$$

Noticing ${}^C D^{\alpha_i} t^{\alpha_i - m} = 0, m = 1, 2, \ldots, N$, where N is the smallest integer greater than or equal to α_i. So, we have

$$\begin{aligned}
{}^C D^{\alpha_1} u(t) &= {}^C D^{\alpha_1}[I_{0+}^{\alpha_1} f_1(t, u(t), v(t), {}^C D^{\rho_1} u(t)), {}^C D^{\rho_2} v(t))] \\
&= f_1(t, u(t), v(t), {}^C D^{\rho_1} u(t)), {}^C D^{\rho_2} v(t)), \\
{}^C D^{\alpha_2} v(t) &= {}^C D^{\alpha_2}[I_{0+}^{\alpha_2} f_2(t, u(t), v(t), {}^C D^{\rho_1} u(t)), {}^C D^{\rho_2} v(t))] \\
&= f_2(t, u(t), v(t), {}^C D^{\rho_1} u(t)), {}^C D^{\rho_2} v(t)).
\end{aligned}$$

By direct computation, we can verify easily that $u(\xi_1) = 0, u(1) = I_{0+}^{\theta_1} u(\eta_1), v(\xi_2) = 0, v(1) = I_{0+}^{\theta_2} v(\eta_2)$. Therefore, $(u, v) \in U_{\alpha_1} \times U_{\alpha_2}$ is a solution of FDE (1.1). The proof is completed. □

Lemma 2.5 [see [27]] (Leray–Schauder nonlinear alternative) *Let F be a Banach space and Ω be a bounded open subset of F, $0 \in \Omega$, $T : \overline{\Omega} \to F$ be a completely continuous operator. Then, either there exists $x \in \partial\Omega$, $\lambda > 1$ such that $T(x) = \lambda x$, or there exists a fixed point $x^* \in \overline{\Omega}$.*

3 Main results

In the following subsection, we establish our main results for FDE (1.1) by using a variety of fixed point theorems. For convenience, we set

$$\begin{aligned}
A_i &= \frac{|\Delta_{i3}|(\xi_i + 1) + |\Delta_{i3}|(|\Delta_{i1}| + |\Delta_{i2}|)\xi_i^{\alpha_i} + 1}{\Gamma(\alpha_i + 1)} + \frac{|\Delta_{i3}|(\xi_i + 1)\eta_i^{\alpha_i + \theta_i}}{\Gamma(\alpha_i + \theta_i + 1)}, \ i = 1, 2, \\
B_i &= \frac{1}{\Gamma(2 - \rho_i)}\left(\frac{|\Delta_{i3}| + |\Delta_{i3}\Delta_{i1}|\xi_i^{\alpha_i}}{\Gamma(\alpha_i + 1)} + \frac{1}{\Gamma(\alpha_i)} + \frac{|\Delta_{i3}|\eta_i^{\alpha_i + \theta_i}}{\Gamma(\alpha_i + \theta_i + 1)}\right), \ i = 1, 2, \\
C_{ik} &= \Big(A_i + B_i\Big)a_{ik}, \ i = 1, 2, k = 1, 2, 3, 4, 5, \\
D_{ik} &= \Big(A_i + B_i\Big)b_{ik}, \ i = 1, 2, k = 1, 2, 3, 4, 5, \\
E_i &= \max\Big\{D_{i1}, D_{i2}, D_{i3}, D_{i4}, D_{i5}\Big\}, i = 1, 2, \\
F_{ik} &= \Big(A_i + B_i\Big)c_{ik}, \ i = 1, 2, k = 1, 2, 3, 4.
\end{aligned}$$

The first result is based on the Schauder fixed point theorem.

Theorem 3.1 *Assume that there exist positive constants $a_{ik} \in (0, +\infty)(i = 1, 2, k = 1, 2, 3, 4, 5)$ such that the following condition is satisfied*

$$(H_1) \ \ |f_i(t, x_1, x_2, x_3, x_4)| \le \sum_{k=1}^{4} a_{ik}|x_k|^{\tau_{ik}} + a_{i5}, 0 < \tau_{ik} < 1.$$

Then FDE (1.1) has at least one solution.

Proof First, define a ball in Banach space $X \times Y$ as

$$B_R = \{(u, v)|(u, v) \in X \times Y, \|(u, v)\|_{X\times Y} \le R\}, \tag{3.1}$$

where

$$R \ge \max\left\{(10C_{i1})^{\frac{1}{1-\tau_{i1}}}, (10C_{i2})^{\frac{1}{1-\tau_{i2}}}, (10C_{i3})^{\frac{1}{1-\tau_{i3}}}, (10C_{i4})^{\frac{1}{1-\tau_{i4}}}, 10C_{i5}, i = 1, 2\right\}.$$

Now we prove that $T : B_R \to B_R$. For any $(u, v) \in B_R$, applying Definition 2.2 and the relation condition (H_1), we have

$$\begin{aligned}
&|T_1(u, v)(t)| \\
&\le |\Delta_{13}(\xi_1 - t)| I_{0+}^{\alpha_1} |f_1(1, u(1), v(1), {}^C D^{\rho_1} u(1), {}^C D^{\rho_2} v(1))| \\
&\quad + |\Delta_{13}(t - \xi_1)| I_{0+}^{\alpha_1+\theta_1} |f_1(\eta_1, u(\eta_1), v(\eta_1), {}^C D^{\rho_1} u(\eta_1), {}^C D^{\rho_2} v(\eta_1))| \\
&\quad + |\Delta_{13}(\Delta_{11} t - \Delta_{12})| I_{0+}^{\alpha_1} |f_1(\xi_1, u(\xi_1), v(\xi_1), {}^C D^{\rho_1} u(\xi_1), {}^C D^{\rho_2} v(\xi_1))| \\
&\quad + I_{0+}^{\alpha_1} |f_1(t, u(t), v(t), {}^C D^{\rho_1} u(t), {}^C D^{\rho_2} v(t))| \\
&\le \left[\frac{|\Delta_{13}|(\xi_1 + 1)}{\Gamma(\alpha_1)} \int_0^1 (1 - s)^{\alpha_1 - 1} \mathrm{d}s + \frac{|\Delta_{13}|(\xi_1 + 1)}{\Gamma(\alpha_1 + \theta_1)} \int_0^{\eta_1} (\eta_1 - s)^{\alpha_1+\theta_1-1} \mathrm{d}s \right. \\
&\quad \left. + \frac{|\Delta_{13}|(|\Delta_{11}| + |\Delta_{12}|)}{\Gamma(\alpha_1)} \int_0^{\xi_1} (\xi_1 - s)^{\alpha_1-1} \mathrm{d}s + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t - s)^{\alpha_1-1} \mathrm{d}s \right] \\
&\quad \times \left(\sum_{k=1}^{4} a_{1k} R^{\tau_{1k}} + a_{15}\right) \\
&\le A_1 \left(\sum_{k=1}^{4} a_{1k} R^{\tau_{1k}} + a_{15}\right).
\end{aligned} \tag{3.2}$$

On the other hand, we have

$${}^C D^{\rho_1} T_1(u, v)(t) = \frac{1}{\Gamma(1 - \rho_1)} \int_0^t \frac{T_1(u, v)'(s)}{(t - s)^{\rho_1}} ds. \tag{3.3}$$

By Remak 2.1, using similar computation as getting (3.2), we have

$$\begin{aligned}
&|T_1(u, v)'(t)| \\
&\le \left[\frac{|\Delta_{13}|}{\Gamma(\alpha_1)} \int_0^1 (1 - s)^{\alpha_1-1} ds + \frac{|\Delta_{13}|}{\Gamma(\alpha_1 + \theta_1)} \int_0^{\eta_1} (\eta_1 - s)^{\alpha_1+\theta_1-1} ds \right. \\
&\quad \left. + \frac{|\Delta_{13}\Delta_{11}|}{\Gamma(\alpha_1)} \int_0^{\xi_1} (\xi_1 - s)^{\alpha_1-1} ds + \frac{1}{\Gamma(\alpha_1 - 1)} \int_0^t (t - s)^{\alpha_1-2} ds \right] \\
&\quad \times \left(\sum_{k=1}^{4} a_{1k} R^{\tau_{1k}} + a_{15}\right)
\end{aligned}$$

$$\leq \left(\frac{|\Delta_{13}| + |\Delta_{13}\Delta_{11}|\xi_1^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{1}{\Gamma(\alpha_1)} + \frac{|\Delta_{13}|\eta_1^{\alpha_1+\theta_1}}{\Gamma(\alpha_1+\theta_1+1)}\right) \times \left(\sum_{k=1}^{4} a_{1k}R^{\tau_{1k}} + a_{15}\right). \tag{3.4}$$

Consequently by (3.3) and (3.4), we have

$$\begin{aligned} |^C D^{\rho_1} T_1(u,v)(t)| \leq & \frac{1}{\Gamma(1-\rho_1)} \int_0^t (t-s)^{-\rho_1} ds \times \left(\frac{|\Delta_{13}| + |\Delta_{13}\Delta_{11}|\xi_1^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{1}{\Gamma(\alpha_1)}\right. \\ & \left. + \frac{|\Delta_{13}|\eta_1^{\alpha_1+\theta_1}}{\Gamma(\alpha_1+\theta_1+1)}\right) \times \left(\sum_{k=1}^{4} a_{1k}R^{\tau_{1k}} + a_{15}\right) \\ \leq & B_1\left(\sum_{k=1}^{4} a_{1k}R^{\tau_{1k}} + a_{15}\right). \end{aligned} \tag{3.5}$$

From (3.2) and (3.5), we have

$$||T_1(u,v)||_X \leq \sum_{k=1}^{4} C_{1k}R^{\tau_{1k}} + C_{15} \leq \frac{1}{10}R + \frac{1}{10}R + \frac{1}{10}R + \frac{1}{10}R + \frac{1}{10}R = \frac{R}{2}.$$

Similarly, one can obtain

$$||T_2(u,v)||_Y \leq \sum_{k=1}^{4} C_{2k}R^{\tau_{2k}} + C_{25} \leq \frac{1}{10}R + \frac{1}{10}R + \frac{1}{10}R + \frac{1}{10}R + \frac{1}{10}R = \frac{R}{2}.$$

That is

$$||T(u,v)||_{X\times Y} = ||T_1(u,v)||_X + ||T_2(u,v)||_Y \leq R.$$

Thus, we have $T : B_R \to B_R$.

Notice that $T_1(u,v)(t)$, $T_2(u,v)(t)$, $^C D^{\rho_1} T_1(u,v)(t)$, $^C D^{\rho_2} T_2(u,v)(t)$ are continuous on [0, 1]. Thus, operator T is also continuous.

Now we show that T is equicontinuous. For this we fixed

$$M_i = \max_{t\in[0,1]} \left\{|f_i(t, u(t), v(t), ^C D_{0^+}^{\rho_1} u(t)), ^C D_{0^+}^{\rho_2} v(t))|\right\}, i = 1, 2.$$

for any $(u,v) \in B_R$. Let $t_1, t_2 \in [0,1] (t_1 < t_2)$, we have

$$\begin{aligned} & |T_1(u,v)(t_2) - T_1(u,v)(t_1)| \\ \leq & M_1\left[\frac{|\Delta_{13}|(t_2-t_1)}{\Gamma(\alpha_1)} \int_0^1 (1-s)^{\alpha_1-1} ds + \frac{|\Delta_{13}|(t_2-t_1)}{\Gamma(\alpha_1+\theta_1)} \int_0^{\eta_1} (\eta_1 - s)^{\alpha_1+\theta_1-1} ds \right. \\ & \left. + \frac{|\Delta_{13}\Delta_{11}|(t_2-t_1)}{\Gamma(\alpha_1)} \int_0^{\xi_1} (\xi_1 - s)^{\alpha_1-1} ds\right] \end{aligned}$$

$$+M_1\left[\frac{1}{\Gamma(\alpha_1)}\int_0^{t_1}[(t_2-s)^{\alpha_1-1}-(t_1-s)^{\alpha_1-1}]ds+\frac{1}{\Gamma(\alpha_1)}\int_{t_1}^{t_2}(t_2-s)^{\alpha_1-1}ds\right]$$
$$\le M_1|\Delta_{13}|\left[\frac{1+\xi_1^{\alpha_1}|\Delta_{11}|}{\Gamma(\alpha_1+1)}+\frac{\eta_1^{\alpha_1+\theta_1}}{\Gamma(\alpha_1+\theta_1+1)}\right](t_2-t_1)+\frac{M_1}{\Gamma(\alpha_1+1)}(t_2^{\alpha_1}-t_1^{\alpha_1}). \tag{3.6}$$

On the other hand, we have

$$\begin{aligned}
&|{}^C D^{\rho_1}T_1(u,v)(t_2)-{}^C D^{\rho_1}T_1(u,v)(t_1)|\\
&=\frac{1}{\Gamma(1-\rho_1)}\left|\int_0^{t_2}\frac{T_1(u,v)'(s)}{(t_2-s)^{\rho_1}}ds-\int_0^{t_1}\frac{T_1(u,v)'(s)}{(t_1-s)^{\rho_1}}ds\right|\\
&\le\frac{1}{\Gamma(1-\rho_1)}\left[\int_0^{t_1}\frac{(t_2-s)^{\rho_1}-(t_1-s)^{\rho_1}}{(t_1-s)^{\rho_1}(t_2-s)^{\rho_1}}|T_1(u,v)'(s)|ds+\int_{t_1}^{t_2}\frac{|T_1(u,v)'(s)|}{(t_2-s)^{\rho_1}}ds\right],
\end{aligned} \tag{3.7}$$

where

$$|T_1(u,v)'(s)|\le\left(\frac{|\Delta_{13}|+|\Delta_{13}\Delta_{11}|\xi_1^{\alpha_1}}{\Gamma(\alpha_1+1)}+\frac{1}{\Gamma(\alpha_1)}+\frac{|\Delta_{13}|\eta_1^{\alpha_1+\theta_1}}{\Gamma(\alpha_1+\theta_1+1)}\right)M_1.$$

From (3.6) and (3.7), we have

$$\begin{aligned}
&|{}^C D^{\rho_1}T_1(u,v)(t_2)-{}^C D^{\rho_1}T_1(u,v)(t_1)|\\
&\le\left(\frac{|\Delta_{13}|+|\Delta_{13}\Delta_{11}|\xi_1^{\alpha_1}}{\Gamma(\alpha_1+1)}+\frac{1}{\Gamma(\alpha_1)}+\frac{|\Delta_{13}|\eta_1^{\alpha_1+\theta_1}}{\Gamma(\alpha_1+\theta_1+1)}\right)\frac{M_1}{\Gamma(1-\rho_1)}\\
&\times\left[\int_0^{t_1}\frac{(t_2-s)^{\rho_1}-(t_1-s)^{\rho_1}}{(t_2-s)^{\rho_1}(t_1-s)^{\rho_1}}ds+\int_{t_1}^{t_2}\frac{1}{(t_2-s)^{\rho_1}}ds\right]\\
&\le B_1M_1\left[2(t_2-t_1)^{1-\rho_1}+t_2^{1-\rho_1}-t_1^{1-\rho_1}\right].
\end{aligned} \tag{3.8}$$

Analogously, one can prove that

$$\begin{aligned}
|T_2(u,v)(t_2)-T_2(u,v)(t_1)|\le M_2|\Delta_{23}|&\left[\frac{1+\xi_2^{\alpha_2}|\Delta_{21}|}{\Gamma(\alpha_2+1)}+\frac{\eta_2^{\alpha_2+\theta_2}}{\Gamma(\alpha_2+\theta_2+1)}\right]\\
&\times(t_2-t_1)+\frac{M_2}{\Gamma(\alpha_2+1)}(t_2^{\alpha_2}-t_1^{\alpha_2}),
\end{aligned} \tag{3.9}$$

$$|{}^C D^{\rho_2}T_2(u,v)(t_2)-{}^C D^{\rho_2}T_2(u,v)(t_1)|\le B_2M_2\left[2(t_2-t_1)^{1-\rho_2}+t_2^{1-\rho_2}-t_1^{1-\rho_2}\right]. \tag{3.10}$$

In (3.6), (3.8), (3.9) and (3.10), letting $t_1\to t_2$, then

$$|T_1(u,v)(t_2)-T_1(u,v)(t_1)|\to 0,\ |{}^C D^{\rho_1}T_1(u,v)(t_2)-{}^C D^{\rho_1}T_1(u,v)(t_1)|\to 0,$$
$$|T_2(u,v)(t_2)-T_2(u,v)(t_1)|\to 0,\ |{}^C D^{\rho_2}T_2(u,v)(t_2)-{}^C D^{\rho_2}T_2(u,v)(t_1)|\to 0.$$

So

$$||T_1(u,v)(t_2) - T_1(u,v)(t_1)||_X \to 0,\ ||T_2(u,v)(t_2) - T_2(u,v)(t_1)||_Y \to 0.$$

That is, as $t_1 \to t_2$,

$$||T(u,v)(t_2) - T(u,v)(t_1)||_{X\times Y} \to 0.$$

Therefore it follows from the above proof that $T(B_R)$ is an equicontinuous set. Also, it is uniformly bounded as $T(B_R) \subset B_R$. By means of the Arzelà–Ascoli theorem, we conclude that T is a completely continuous operator. Hence, applying the Schauder fixed theorem, FDE (1.1) has at least one solution (u, v) in B_R. This proof is completed. □

Remark 3.1 The condition (H_1) can be replaced by the following condition

$$(H_2)\ \ |f_i(t, x_1, x_2, x_3, x_4)| \le \sum_{k=1}^{4} a_{ik}|x_k|^{\tau_{ik}},\ \tau_{ik} > 1.$$

and the conclusion of Theorem 3.1 remains true. Noticing, some additional restriction about R in (3.1) should be replaced by the following restriction

$$0 < R < \min\left\{\left(\frac{1}{8C_{i1}}\right)^{\frac{1}{\tau_{i1}-1}}, \left(\frac{1}{8C_{i2}}\right)^{\frac{1}{\tau_{i2}-1}}, \left(\frac{1}{8C_{i3}}\right)^{\frac{1}{\tau_{i3}-1}}, \left(\frac{1}{8C_{i4}}\right)^{\frac{1}{\tau_{i4}-1}}, i = 1, 2\right\}.$$

So, repeating arguments similar to proof of Theorem 3.1, we can obtain the same conclusion.

The second result is based on the Leray–Schauder nonlinear alternative.

Theorem 3.2 *Assume that there exist positive constants* $b_{ik} \in (0, +\infty))(i = 1, 2, k = 1, 2, 3, 4, 5)$ *and functions* $\phi_{ik} \in C([0, +\infty), (0, +\infty))(i = 1, 2, k = 1, 2, 3, 4)$ *nondecreasing on* $[0, +\infty)$ *and* $L > 0$ *such that the following conditions are satisfied*

$$(H_3)\ \ |f_i(t, x_1, x_2, x_3, x_4)| \le \sum_{k=1}^{4} b_{ik}\phi_{ik}(|x_k|) + b_{i5}, i = 1, 2,$$

$$(H_4)\ \ E_1\Big(\sum_{k=1}^{4} \phi_{1k}(L) + 1\Big) + E_2\Big(\sum_{k=1}^{4} \phi_{2k}(L) + 1\Big) < L.$$

Then FDE (1.1) has at least one solution.

Proof First we show that T is completely continuous. It is obvious that T is continuous since f_i are continuous. For a positive number $L > 0$, let $B_L = \{(u, v)|(u, v) \in$

$X \times Y, \|(u, v)\|_{X\times Y} \leq L\}$ be a bounded ball in $X \times Y$. We shall prove that $T(B_L)$ is relatively compact.

For $(u, v) \in T(B_L)$, similar computation as (3.2) and (3.5) yields

$$|T_1(u, v)(t)| \leq A_1\left(\sum_{k=1}^{4} b_{1k}\phi_{1k}(L) + b_{15}\right), \tag{3.11}$$

$$|{}^C D^{\rho_1} T_1(u, v)(t)| \leq B_1\left(\sum_{k=1}^{4} b_{1k}\phi_{1k}(L) + b_{15}\right). \tag{3.12}$$

Hence

$$||T_1(u, v)||_X \leq (A_1 + B_1)\left(\sum_{k=1}^{4} b_{1k}\phi_{1k}(L) + b_{15}\right) = \sum_{k=1}^{4} D_{1k}\phi_{1k}(L) + D_{15}. \tag{3.13}$$

Similarly, one can obtain

$$||T_2(u, v)||_Y \leq \sum_{k=1}^{4} D_{2k}\phi_{2k}(L) + D_{25}. \tag{3.14}$$

Combining (3.13) with (3.14), we get

$$||T(u, v)||_{X\times Y} = ||T_1(u, v)||_X + ||T_2(u, v)||_Y \leq \left(\sum_{k=1}^{4} D_{1k}\phi_{1k}(L) + D_{15}\right) + \left(\sum_{k=1}^{4} D_{2k}\phi_{2k}(L) + D_{25}\right). \tag{3.15}$$

Hence, $T(B_L)$ is uniformly bounded through (3.15). Next as the similar computation as (3.6)–(3.10) yields that $T(B_L)$ is an equicontinuous set. By means of the Arzelà–Ascoli theorem, we conclude that T is a completely continuous operator.

Now we apply the Leray–Schauder nonlinear alternative (Lemma 2.5) to prove that T has at least one solution in $X \times Y$.

For $(u, v) \in \partial B_L$, such that $(u, v) = \lambda T(u, v)$, $0 < \lambda < 1$. By (3.11) and (3.12), we have

$$|u(t)| = \lambda|T_1(u, v)(t)| \leq |T_1(u, v)(t)| \leq A_1\left[\sum_{k=1}^{4} b_{1k}\phi_{1k}(L) + b_{15}\right],$$

$$|{}^C D^{\rho_1}(u, v)(t)| = \lambda|{}^C D^{\rho_1} T_1(u, v)(t)| \leq |{}^C D^{\rho_1} T_1(u, v)(t)| \leq B_1\left[\sum_{k=1}^{4} b_{1k}\phi_{1k}(L) + b_{15}\right].$$

Hence

$$\begin{aligned} ||u||_X &\le (A_1 + B_1)\left[\sum_{k=1}^{4} b_{1k}\phi_{1k}(L) + b_{15}\right] = \sum_{k=1}^{4} D_{1k}\phi_{1k}(L) + D_{15} \\ &\le E_1\left[\sum_{k=1}^{4}\phi_{1k}(L) + 1\right]. \end{aligned} \tag{3.16}$$

Similarly, one can obtain

$$||v||_Y \le E_2\left[\sum_{k=1}^{4}\phi_{2k}(L) + 1\right]. \tag{3.17}$$

Combining (3.16), (3.17) with the condition (H_4), we get

$$\|(u, v)\|_{X\times Y} = ||u||_X + ||v||_Y \le E_1\left(\sum_{k=1}^{4}\phi_{1k}(L) + 1\right) + E_2\left(\sum_{k=1}^{4}\phi_{2k}(L) + 1\right) < L, \tag{3.18}$$

this contradicts the fact $(u, v) \in \partial B_L$. By Lemma 2.5 we conclude that T has a fixed point $(u, v) \in \overline{B}_L$ and then FDE (1.1) has at least one solution in $X \times Y$. □

Corollary 3.1 *Assume that there exist positive constants* $b_{ik} \in (0, +\infty))(i = 1, 2, k = 1, 2, 3, 4, 5)$ *and* $L > 0$ *such that the following conditions are satisfied*

$$(H_5)\ \ |f_i(t, x_1, x_2, x_3, x_4)| \le \sum_{k=1}^{4} b_{ik}|x_k| + b_{i5},\ i = 1, 2,$$

$$(H_6)\ \ (E_1 + E_2)(4L + 1) < L.$$

Then FDE (1.1) has at least one solution.

Remark 3.2 We obtained the existence of solutions for nonlinear FDE (1.1) by Theorem 3.1 and Theorem 3.2. Some growth conditions

$$|f_i(t, x_1, x_2, x_3, x_4)| \le \sum_{k=1}^{4} a_{ik}|x_k|^{\tau_{ik}} + a_{i5}$$

is given through three cases: In (H_1), $0 < \tau_{ik} < 1$; In (H_2), $\tau_{ik} > 1$, for the sake of simplicity, here $a_{i5} = 0$; In (H_5), $\tau_{ik} = 1$, and some additional restriction (H_6) is given. Obviously, it is easy to know that the conclusion of Theorem 3.2 contains result of Theorem 3.1, but the condition of Theorem 3.1 is verified easily and more convenient to apply, see Example 4.1.

The uniqueness of solutions is based on the Banach contraction principle.

Theorem 3.3 *Assume that there exist positive constants* $c_{ik} \in (0, +\infty))(i = 1, 2, k = 1, 2, 3, 4)$ *such that the following conditions are satisfied, for all* $t \in [0, 1]$ *and* $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 \in \mathbb{R}$,

$$(H_7)\ \ |f_i(t, x_1, x_2, x_3, x_4) - f_i(t, y_1, y_2, y_3, y_4)| \leq \sum_{k=1}^{4} c_{ik}|x_k - y_k|, i = 1, 2,$$

$$(H_8)\ \ H = \sum_{k=1}^{4}(F_{1k} + F_{2k}) < 1.$$

Then FDE (1.1) has a unique solution.

Proof Define $\sup\limits_{t\in[0,1]} f_i(t, 0, 0, 0, 0) = G_i < \infty, i = 1, 2$ and take

$$r \geq \frac{G'_1 + G'_2}{1 - H},$$

where $G'_i = \Big(A_i + B_i\Big)G_i, i = 1, 2.$

First, we show that $T(B_r) \subset B_r$, where $B_r = \{(u, v)|(u, v) \in X \times Y : ||(u, v)||_{X\times Y} \leq r\}$. For $(u, v) \in B_r$,we have

$$\begin{aligned}
&|T_1(u, v)(t)| \\
&\leq |\Delta_{13}|(\xi_1 + 1)I_{0^+}^{\alpha_1}\Big[|f_1(1, u(1), v(1),^C D^{\rho_1}u(1),^C D^{\rho_2}v(1)) \\
&\quad -f_1(1, 0, 0, 0, 0)| + |f_1(1, 0, 0, 0, 0)|\Big] \\
&\quad +|\Delta_{13}|(\xi_1 + 1)I_{0^+}^{\alpha_1+\theta_1}\Big[|f_1(\eta_1, u(\eta_1), v(\eta_1),^C D^{\rho_1}u(\eta_1)),^C D^{\rho_2}v(\eta_1)) \\
&\quad -f_1(\eta_1, 0, 0, 0, 0)| \\
&\quad +|f_1(\eta_1, 0, 0)|\Big] + |\Delta_{13}|(|\Delta_{11}| \\
&\quad +|\Delta_{12}|)I_{0^+}^{\alpha_1}\Big[|f_1(\xi_1, u(\xi_1), v(\xi_1),^C D^{\rho_1}u(\xi_1)),^C D^{\rho_2}v(\xi_1)) \\
&\quad -f_1(\xi_1, 0, 0, 0, 0)| + |f_1(\xi_1, 0, 0, 0, 0)|\Big] \\
&\quad +I_{0^+}^{\alpha_1}\Big[|f_1(t, u(t), v(t),^C D^{\rho_1}u(t),^C D^{\rho_2}u(t)) \\
&\quad -f_1(t, 0, 0, 0, 0)| + |f_1(t, 0, 0, 0, 0)|\Big] \\
&\leq \Bigg[\frac{|\Delta_{13}|(\xi_1 + 1)}{\Gamma(\alpha_1)}\int_0^1 (1 - s)^{\alpha_1-1}ds + \frac{|\Delta_{13}|(\xi_1 + 1)}{\Gamma(\alpha_1 + \theta_1)}\int_0^{\eta_1} (\eta_1 - s)^{\alpha_1+\theta_1-1}ds \\
&\quad +\frac{|\Delta_{13}|(|\Delta_{11}| + |\Delta_{12}|)}{\Gamma(\alpha_1)}\int_0^{\xi_1} (\xi_1 - s)^{\alpha_1-1}ds + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t - s)^{\alpha_1-1}ds\Bigg]
\end{aligned}$$

$$\times\Big[(c_{11}+c_{13})||u||_X+(c_{12}+c_{14})||v||_Y+G_1\Big]$$
$$\le A_1\Big[(c_{11}+c_{13})||u||_X+(c_{12}+c_{14})||v||_Y+G_1\Big]. \tag{3.19}$$

On the other hand, we have

$$|T_1(u,v)'(t)|\le\left(\frac{|\Delta_{13}|+|\Delta_{13}\Delta_{11}|\xi_1^{\alpha_1}}{\Gamma(\alpha_1+1)}+\frac{1}{\Gamma(\alpha_1)}+\frac{|\Delta_{13}|\eta_1^{\alpha_1+\theta_1}}{\Gamma(\alpha_1+\theta_1+1)}\right)$$
$$\times\Big[(c_{11}+c_{13})||u||_X+(c_{12}+c_{14})||v||_Y+G_1\Big]. \tag{3.20}$$

Using (3.3) and (3.20), we obtain

$$|^C D^{\rho_1}T_1(u,v)(t)|\le B_1\Big[(c_{11}+c_{13})||u||_X+(c_{12}+c_{14})||v||_Y+G_1\Big]. \tag{3.21}$$

Combining (3.19) and (3.21), we get

$$||T_1(u,v)||_X\le(F_{11}+F_{13})||u||_X+(F_{12}+F_{14})||v||_Y+G'_1\le r\sum_{k=1}^{4}F_{1k}+G'_1.$$

Similarly, one has

$$||T_2(u,v)||_Y\le(F_{21}+F_{23})||u||_X+(F_{22}+F_{24})||v||_Y+G'_2\le r\sum_{k=1}^{4}F_{2k}+G'_2.$$

Consequently

$$||T(u,v)||_{X\times Y}=||T_1(u,v)||_X+||T_2(u,v)||_Y\le Hr+G'_1+G'_2\le r.$$

Now for any $(u_2,v_2),(u_1,v_1)\in B_r$,we have

$$|T_1(u_2,v_2)(t)-T_1(u_1,v_1)(t)|$$
$$\le\Bigg[\frac{|\Delta_{13}|(\xi_1+1)}{\Gamma(\alpha_1)}\int_0^1(1-s)^{\alpha_1-1}ds+\frac{|\Delta_{13}|(\xi_1+1)}{\Gamma(\alpha_1+\theta_1)}\int_0^{\eta_1}(\eta_1-s)^{\alpha_1+\theta_1-1}ds$$
$$+\frac{|\Delta_{13}|(|\Delta_{11}|+|\Delta_{12}|)}{\Gamma(\alpha_1)}\int_0^{\xi_1}(\xi_1-s)^{\alpha_1-1}ds+\frac{1}{\Gamma(\alpha_1)}\int_0^t(t-s)^{\alpha_1-1}ds\Bigg]$$
$$\times\Big[(c_{11}+c_{13})||u_2-u_1||_X+(c_{12}+c_{14})||v_2-v_1||_Y\Big]$$
$$\le A_1\Big[(c_{11}+c_{13})||u_2-u_1||_X+(c_{12}+c_{14})||v_2-v_1||_Y\Big], \tag{3.22}$$

and

$$\begin{aligned} &|^C D^{\rho_2} T_1(u_2, v_2)(t) -^C D^{\rho_2} T_1(u_1, v_1)(t)| \\ &\leq B_1\Big[(c_{11} + c_{13})||u_2 - u_1||_X + (c_{12} + c_{14})||v_2 - v_1||_Y\Big]. \end{aligned} \tag{3.23}$$

Combining (3.22) and (3.23), we get

$$||T_1(u_2, v_2) - T_1(u_1, v_1)||_X \leq (F_{11} + F_{13})||u_2 - u_1||_X + (F_{12} + F_{14})||v_2 - v_1||_Y.$$

Similarly, one has

$$||T_2(u_2, v_2) - T_2(u_1, v_1)||_Y \leq (F_{21} + F_{23})||u_2 - u_1||_X + (F_{22} + F_{24})||v_2 - v_1||_Y.$$

Therefore

$$||T(u_2, v_2) - T(u_1, v_1)||_{X\times Y} \leq H\Big[||u_2 - u_1||_X + ||v_2 - v_1||_Y\Big].$$

Since $H < 1$, T is a contraction operator. So, using the Banach contraction principle, the operator T has a unique fixed point, which is the unique solution of FDE (1.1). □

4 Some examples

In this section, in order to illustrate our results, we consider the following three examples.

Example 4.1 Consider the following coupled system of nonlinear FDE with fractional integral conditions

$$\begin{cases} {}^C D_{0+}^{\frac{3}{2}} u(t) = a_{11}(u(t))^{\tau_{11}} + a_{12}(v(t))^{\tau_{12}} + a_{13}({}^C D_{0+}^{\frac{1}{4}} u(t))^{\tau_{13}} \\ \qquad + a_{14}({}^C D_{0+}^{\frac{1}{3}} v(t))^{\tau_{14}} + a_{15}, t \in (0, 1), \\ {}^C D_{0+}^{\frac{5}{4}} v(t) = a_{21}(u(t))^{\tau_{21}} + a_{22}(v(t))^{\tau_{22}} + a_{23}({}^C D_{0+}^{\frac{1}{4}} u(t))^{\tau_{23}} \\ \qquad + a_{24}({}^C D_{0+}^{\frac{1}{3}} v(t))^{\tau_{24}} + a_{25}, t \in (0, 1), \\ u(\frac{1}{10}) \quad = 0, u(1) = I_{0+}^{\frac{5}{4}} u(\frac{1}{5}), \\ v(\frac{1}{16}) \quad = 0, v(1) = I_{0+}^{\frac{1}{3}} v(\frac{1}{8}), \end{cases} \tag{4.1}$$

where $0 < \tau_{ij} < 1 (i = 1, 2; j = 1, 2, 3, 4)$ and $a_{ik} (i = 1, 2; k = 1, 2, 3, 4, 5)$ are positive constants. Obviously, it follows from Theorem 3.1 that FDE (4.1) has at least one solution.

Example 4.2 Consider the following coupled system of nonlinear FDE with fractional integral conditions

$$\begin{cases} {}^C D_{0+}^{\frac{3}{2}} u(t) = \frac{u^2}{200} + \frac{v^2}{100(1+|v|)} + \frac{1}{50}({}^C D_{0+}^{\frac{1}{2}} u(t)) \\ \qquad\qquad + \frac{t}{500}({}^C D_{0+}^{\frac{1}{2}} v(t))^2 + \frac{(1-t)^2}{100}, t \in (0,1), \\ 7\,{}^C D_{0+}^{\frac{5}{4}} v(t) = \frac{(1-t)u^4}{100(1+u^2)} \\ \qquad\qquad + \frac{v^2}{200} + \frac{t^2}{100}({}^C D_{0+}^{\frac{1}{2}} u(t))^3 + \frac{1}{200}({}^C D_{0+}^{\frac{1}{2}} v(t)) + \frac{(1+t)^2}{400}, t \in (0,1), \\ u(\frac{1}{4}) = 0, u(1) = I_{0+}^{\frac{1}{3}} u(\frac{1}{2}), \\ u(\frac{1}{2}) = 0, u(1) = I_{0+}^{\frac{5}{4}} u(\frac{1}{3}), \end{cases} \tag{4.2}$$

where $\alpha_1 = \frac{3}{2}, \alpha_2 = \frac{5}{4}, \rho_1 = \rho_2 = \frac{1}{2}, \xi_1 = \frac{1}{4}, \xi_2 = \frac{1}{2}, \eta_1 = \frac{1}{2}, \eta_2 = \frac{1}{3}, \theta_1 = \frac{1}{3}, \theta_2 = \frac{5}{4}$. By (4.2), we have

$$\begin{aligned} |f_1(t, x_1, x_2, x_3, x_4)| &= \frac{x_1^2}{200} + \frac{x_2^2}{100(1+|x_2|)} + \frac{x_3}{50} + \frac{t x_4^2}{500} + \frac{(1-t)^2}{100} \\ &\leq \frac{|x_1|^2}{200} + \frac{|x_2|}{100} + \frac{|x_3|}{50} + \frac{|x_4|^2}{500} + \frac{1}{100} = a_{11}\phi_{11}(|x_1|) \\ &\quad + a_{12}\phi_{12}(|x_2|) + a_{13}\phi_{13}(|x_3|) + a_{14}\phi_{14}(|x_4|) + a_{15}, \\ |f_2(t, x_1, x_2, x_3, x_4)| &= \frac{(1-t)x_1^4}{100(1+x_1^2)} + \frac{x_2^2}{200} + \frac{t^2|x_3|^3}{100} + \frac{x_4}{200} + \frac{(1+t)^2}{400} \\ &\leq \frac{x_1^2}{100} + \frac{x_2^2}{200} + \frac{|x_3|^3}{100} + \frac{|x_4|}{200} + \frac{1}{100} = a_{21}\phi_{21}(|x_1|) \\ &\quad + a_{22}\phi_{22}(|x_2|) + a_{23}\phi_{23}(|x_3|) + a_{24}\phi_{24}(|x_4|) + a_{25}, \end{aligned}$$

where $b_{11} = \frac{1}{200}, b_{12} = \frac{1}{100}, b_{13} = \frac{1}{50}, b_{14} = \frac{1}{500}, b_{15} = \frac{1}{100}, \phi_{11}(|x_1|) = |x_1|^2, \phi_{12}(|x_2|) = |x_2|, \phi_{13}(|x_3|) = |x_3|, \phi_{14}(|x_4|) = |x_4|^2, b_{21} = \frac{1}{100}, b_{22} = \frac{1}{200}, b_{23} = \frac{1}{100}, b_{24} = \frac{1}{200}, b_{25} = \frac{1}{100}, \phi_{21}(|x_1|) = |x_1|^2, \phi_{22}(|x_2|) = |x_2|^2, \phi_{23}(|x_3|) = |x_3|^3, \phi_{24}(|x_4|) = |x_4|$. Let us evaluate $[E_1(\sum_{k=1}^{4} \phi_{1k}(L)+1) + E_2(\sum_{k=1}^{4} \phi_{2k}(L)+1) - L]$. By direct calculation, we can obtain that

$$\begin{aligned} \Delta_{11} &= 0.1110, \ \Delta_{12} = 0.6666, \ \Delta_{13} = 1.5653, \\ \Delta_{21} &= 0.7764, \ \Delta_{22} = 0.9669, \ \Delta_{23} = 1.7280, \\ A_1 &= 1.9047, \ A_2 = 3.4556, \ B_1 = 2.9083, \\ B_2 &= 3.5656, \ C_{11} = 0.0241, \ C_{12} = 0.0482, \\ D_{13} &= 0.0964, \ D_{14} = 0.0096, \ D_{15} = 0.0482, \\ D_{21} &= 0.0702, \ D_{22} = 0.0351, \ D_{23} = 0.0702, \\ D_{24} &= 0.0351, \ D_{25} = 0.0702, \ E_1 = 0.0964, \ E_2 = 0.0702. \end{aligned}$$

Then $E_1\big(\sum_{k=1}^{4}\phi_{1k}(L)+1\big)+E_2\big(\sum_{k=1}^{4}\phi_{2k}(L)+1\big)-L=0.0964\times5+0.0702\times5-1=-0.167<0$ for $L=1$. Theorem 3.2 implies that FDE (4.2) has at least one solution.

Example 4.3 Consider the following coupled system of nonlinear FDE with fractional integral conditions

$$\begin{cases} {}^C D_{0^+}^{\frac{3}{2}}u(t)=\frac{1}{40}u(t)+\frac{3}{50}v(t)+\frac{3}{100}\,{}^C D_{0^+}^{\frac{1}{2}}u(t)+\frac{7}{200}\,{}^C D_{0^+}^{\frac{1}{2}}v(t)+\sin t,\ t\in(0,1),\\ {}^C D_{0^+}^{\frac{5}{4}}v(t)=\frac{1}{300}u(t)+\frac{1}{600}v(t)+\frac{3}{200}\,{}^C D_{0^+}^{\frac{1}{2}}u(t)+\frac{3}{100}\,{}^C D_{0^+}^{\frac{1}{2}}v(t)+\frac{t}{10},\ t\in(0,1),\\ u(\frac{1}{4})=0,\ u(1)=I_{0^+}^{\frac{1}{3}}u(\frac{1}{2}),\\ v(\frac{1}{2})=0,\ v(1)=I_{0^+}^{\frac{5}{4}}v(\frac{1}{3}), \end{cases} \tag{4.3}$$

where $\alpha_1=\frac{3}{2},\alpha_2=\frac{5}{4},\rho_1=\rho_2=\frac{1}{2},\xi_1=\frac{1}{4},\xi_2=\frac{1}{2},\eta_1=\frac{1}{2},\eta_2=\frac{1}{3},\theta_1=\frac{1}{3},\theta_2=\frac{5}{4}$. By (4.3), we have

$$f_1(t,x_1,x_2,x_3,x_4)=\frac{1}{40}x_1+\frac{3}{50}x_2+\frac{3}{100}x_3+\frac{7}{200}x_4+\sin t,$$
$$f_2(t,x_1,x_2,x_3,x_4)=\frac{1}{300}x_1+\frac{1}{600}x_2+\frac{3}{200}x_3+\frac{3}{100}x_4+\frac{t}{10},$$

and

$$\begin{aligned}&|f_1(t,x_1,x_2,x_3,x_4)-f_1(t,y_1,y_2,y_3,y_4)|\\ \le&\ \frac{1}{40}|x_1-y_1|+\frac{3}{50}|x_2-y_2|+\frac{3}{100}|x_3-y_3|+\frac{7}{200}|x_4-y_4|,\\ &|f_2(t,x_1,x_2,x_3,x_4)-f_2(t,y_1,y_2,y_3,y_4)|\\ \le&\ \frac{1}{300}|x_1-y_1|+\frac{1}{600}|x_2-y_2|+\frac{3}{200}|x_3-y_3|+\frac{3}{100}|x_4-y_4|.\end{aligned}$$

Combining with the calculation result of (4.2), by direct calculation, we can obtain that

$$F_{11}=0.1001,\ F_{12}=0.2404,\ F_{13}=0.1201,\ F_{14}=0.1401,$$
$$F_{21}=0.0234,\ F_{22}=0.0117,\ F_{23}=0.1053,\ F_{24}=0.2106.$$

Hence, $H=0.9517<1$. Theorem 3.3 implies that FDE (4.3) has a unique solution.

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Conflict of interest The authors declare that they have no competing interests.

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